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THE MECHANICS OF THE EARTH'S
ATMOSPHERE

A COLLECTION OF TRANSLATIONS

BY

CLEVELAND ABBE

THIRD COLLECTION



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1910

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XXIV

ON THE ENERGY OF STORMS

BY MAX MARGULES

[Dated November, 1904. Translated from the appendix to the annual volume for 1903 of the Imperial Central Institute for Meteorology. Vienna, 1905]

GENERAL SUMMARY

When the velocity of the wind is 30 *m./sec.*, the living force or kinetic energy of one kilogram of air is 450 *kg. m.² sec.⁻²* or nearly equivalent to 0.1 calorie. This amount, which is not large, in comparison with the energy corresponding to the quantities of heat that 1 *kg.* of air at the surface of the ground receives and loses during one day, does appear very large when we compare it with the energy of a wind of average velocity, such as 5 *m./sec.*

It is not probable that such a very large proportion of the heat communicated to the air is converted into kinetic energy during stormy weather. We shall now seek for conditions of the atmosphere by virtue of which a sufficient supply of potential energy is stored up to maintain the storm; we shall allow ourselves to be led by experience and will start with the relations between the mechanical and the thermal forms of energy in a gaseous mass.

§(1) A mass of air extending from the ground upward and bounded by a vertical wall (and sometimes even the whole atmosphere) will serve as our closed atmospheric system. For any such system the equation of energy¹ is

$$\delta\bar{K} + \delta\bar{P} + \delta\bar{A} + (\bar{R}) = 0$$

where $\delta\bar{K}$ is the increase in the kinetic energy of the system;

$\delta\bar{P}$ is the corresponding change in the potential energy of position, considering gravitation as the only external force;

$-\delta\bar{A}$ is the work done by the pressural forces;

¹I have taken the liberty of substituting capitals with the superscript dash for the German type used by Margules.—ABBE.

$-(\bar{R})$ is the work done by the frictional forces, or $+(R)$ is the loss of kinetic energy by the action of friction and other resistances.

Of these quantities the first two depend only on the initial stage and final stage, the other two depend on the nature of the motion.

In a closed system the work of the pressural forces [or $-\delta A$] is equal to the whole work done by the expansion of the air. Let (Q) be the increase of heat, and $\delta \bar{I}$ the increase of internal energy then

$$(Q) = \delta \bar{I} - \delta \bar{A}$$

For motions that occur without any general increase of heat (but in which internal exchanges of heat or even external additions and withdrawals that balance each other are allowable), the value of $\delta \bar{A}$ has also this same property, since $\delta \bar{I}$ depends only on the final stage.

The general equation of energy for a closed system as deduced from the preceding considerations,

$$(Q) = \delta (\bar{K} + \bar{P} + \bar{I}) + (\bar{R})$$

tells us that that part of the added heat that does not serve to increase the internal energy, represents the increase in kinetic energy and in the potential energy of position and in the consumption of energy in overcoming friction. If there be no increase of heat then the increase of mechanical energy takes place at the expense of the internal energy already present.

§(2) By the help of this last equation we will first seek for a closed dry air system that without any increase of heat can develop such great kinetic energy as we observe in storms.

Let the air be initially at rest but not in equilibrium. It starts in motion and tends to attain a condition of stable equilibrium. In general we know the characteristics of this final condition: every horizontal layer is a surface of equal pressure and equal temperature, the entropy (or the potential temperature) increases with the altitude. In order to completely determine this final stage we will assume that every part of the mass behaves adiabatically, or isentropically, during the motion. We now construct the final stage by the following process: we seek first the masses having the least entropy at the initial stage; these will form the lowest stratum, the other masses will arrange themselves proceeding upward, i

the order of the increasing values of initial entropy. Equal masses of equal entropy can be interchanged at will.

The available kinetic energy of the system, including the loss by friction, is determined by the relation

$$\delta \bar{K} + (\bar{K}) = -\delta(\bar{P} + \bar{I}) = (\bar{P} + \bar{I})_a - (\bar{P} + \bar{I})_e$$

where the index a belongs to the initial and e to the final stage. $(\bar{P} + \bar{I})_e$ is the smallest value that the sum total of potential and internal energy can have under adiabatic condition.

If there be no friction then this equation is to be understood as follows: The largest amount of kinetic energy will be attained when all the masses pass through their respective appropriate positions of equilibrium simultaneously; if this does not occur then the kinetic energy attainable will be less than this which we will designate as "available energy." During the pendulous oscillation of the masses, a part of the potential and of the internal energy is still latent.²

If there be friction then the kinetic energy increases and becomes a larger fraction of the total amount of available energy in proportion as the influence of the friction is smaller; after the masses of air have approached their final positions this fraction diminishes and becomes zero when the final stage is attained. The condition of isentropic change is not precisely fulfilled since the friction produces heat. In our analysis we assume that this heat is again withdrawn from the air-mass. This limitation is of slight importance in the case of atmospheric motions.

§(3) Of all the different kinds of storms those best known to me are the gusts of wind (the *boe-en*) which are accompanied by rapid increase of atmospheric pressure and rapid fall of temperature. These were first investigated by Koeppen. Masses of air of unequal temperatures at identical levels are separated by a sharp boundary that advances with the storm wind toward the warmer side. The difference of temperature, which is often 10° C. at the surface of the ground, continues up to an altitude of nearly 2000 meters. The pressures at greater altitudes, not far from the boundary, are equal over the warm and cold regions but at the ground they are greater in the cold region.

Based on this general experience I have formulated the following problem: Let the mass of air in the lower part of a closed system, A, B, C, D , be initially divided by a screen into two parts

²Helmholtz describes the free (*freie*) energy, F , of any system, the total (*gesammt*) energy, u , and the latent (*gebundene*) energy, $u - F$.—EDITOR.

(see fig. 1a). Let the cold air be in the left hand chamber 1 but the warm air in the right hand 2; each mass of air to be at rest and in either stable or neutral equilibrium; the whole mass of air in the enclosure above these two chambers takes no part in the following processes, we can replace it by a movable heavy piston. What amount of kinetic energy becomes available when we remove the screen and let the masses 1 and 2 move adiabatically?

If in the initial stage the entropy of a kilogram of the highest layer of the mass (1) is smaller than that of the lowest layer of mass (2), then in the final stage the whole cold mass of air will spread out below or at (1'), and the warm mass will be spread out above or at (2') as shown in fig. 1b.

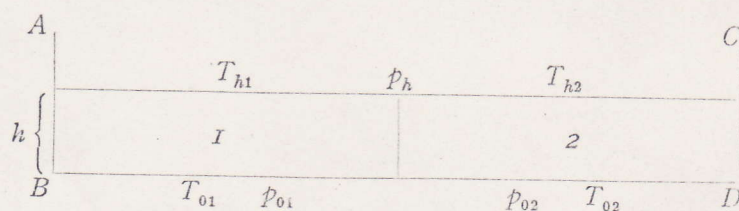


FIG. 1a.

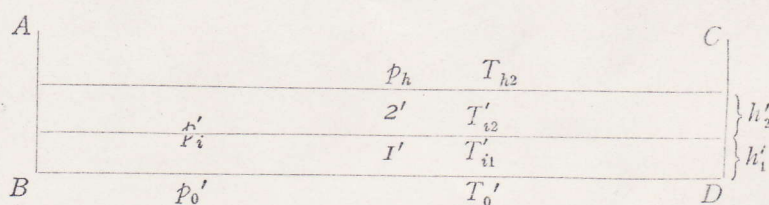


FIG. 1b.

If the equilibrium was stable on both sides of the screen then in every part the serial order of the strata is retained after the overturning.

If the equilibrium in (1) and (2) was neutral in the initial stage then it will remain so in the final stage (1') and (2').

Let T be the absolute temperature of an elementary mass dm in the initial stage; T' that of the same mass in the final stage; C_p the specific heat of air under constant pressure; C_v that for constant volume, $C_p - C_v = R$, then we find

$$\delta \bar{P} = R \int (T' - T) dm \quad \delta \bar{I} = C_v \int (T' - T) dm$$

whence the available kinetic energy of the system is

$$-\delta (\bar{P} + \bar{I}) = C_p \int (T - T') dm$$

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where $\delta\bar{P}$ is the change in the potential energy of position for the whole system including that of the piston. $\delta\bar{P}$ and $\delta\bar{I}$ are the analogs of the exterior work and the change of internal energy of a small mass of air under constant pressure. These equations hold good in general for any change of location of the masses of air when the pressure remains constant in the superposed movable level surface.³ The integrals are to be extended over all the masses lying below this surface.

The quotient of the available kinetic energy by the mass below the piston gives the average energy $\frac{1}{2}V^2$ per unit of mass. When the above assumed overturning of the masses takes place and the volumes of the two chambers are equal, the corresponding value of V is given by

$$V = \frac{1}{2} \sqrt{g h \tau}$$

where g = acceleration of gravity; h = altitude of the chambers; T_1 and T_2 are the initial average temperatures: $\tau = (T_2 - T_1)/T_1$. Assuming $T_1 = 273^\circ$, $T_2 = 283^\circ$ and $h = 2000, 3000, 6000$ meters successively, we find $V = 13, 16, 23$ meters per second respectively.

In the case of cold and warm rooms in dwellings having the same temperatures as above but $h = 5$ meters we find $V = 0.67$ m./sec.

But this computation tells us nothing as to how the available kinetic energy is distributed within the masses. We see, however, that with chambers 2000 meters high and a difference of temperature of 10° C. a storm velocity cannot prevail throughout the whole mass but only in about one-fourth part of it, and in even a still smaller fraction of the whole mass if we abstract a large amount for the loss of the energy due to friction. In boe-en or gusts, strong winds occur only on the cold side and close to the boundary.⁴

The ratio of R to C_p is that of 1 to 2.5 and by this ratio is the work done by the force of gravity less than the diminution of internal energy when the change of location takes place under constant pressure. When the overturning takes place under constant volume we have the same value of V , and so also when it takes place with a very slow change of pressure in the level surfaces near the boundary; but in these cases the ratio $\delta\bar{P}/\delta\bar{I}$ is changed. We have the same available kinetic energy if we fill each chamber with an incompressible fluid whose mass is the same as that of the air in this chamber, but in this case the energy arises from the work of gravity only.

³ Corresponding to the piston.

⁴ So also in northers, blizzards, chinooks, boras, purgas, etc.—C. A

§(4) Let us now consider another system. In the chambers (1) and (2) of fig. 1a let two masses of air be enclosed whose respective entropies are the same at the same altitudes: therefore the greater pressure will be on the side of the warmer air; a difference of pressure of 10^{mm} mercury at the base implies a difference of temperature of 1°C . only. Let each of the two masses be initially in stable equilibrium.

If now the screen be removed and the total volume be unchanged then the horizontal strata that were initially at the same altitudes unite to form one new stratum. The vertical succession remains unchanged and so also the altitude of the center of gravity. The change of internal energy as computed for the unit of mass depends only on the initial difference of pressure, not on the height of the chambers. We may therefore replace this system by a thin horizontal stratum that initially contained air in two equal divisions at pressures of 765^{mm} and 755^{mm} mercury respectively. The available kinetic energy is that appropriate to a velocity of 1.5 m/sec . for the whole mass: that which is available for each unit of mass is far smaller than in those systems that with the same distribution of pressure at the base have horizontal differences of entropy.

It has already been shown by me, in my treatise, "On the work represented by any distribution of atmospheric pressure"⁵ that the energy stored up in such a distribution of pressure as is observed in our lower strata during a stormy period would not suffice to develop the observed kinetic energy of the storm, if the masses of air were only pushed horizontally out of their positions of equilibrium.

A great velocity of a mass of air over a broad area under the influence of a horizontal pressure gradient can only arise when this gradient is maintained by some outside source of energy; otherwise it would disappear before any portion of the mass of air had attained the velocity of a moderate wind. Dry air possesses such a store of energy when horizontal differences of entropy of ordinary amount exist at any level; and not only when there is a sharp boundary between warm and cold air but also when there is a steady continuous horizontal gradient of entropy.

§(5) The available kinetic energy of a system in which masses of unequal entropy are superposed in unstable equilibrium can be computed from the energy equation. Here we find a store of

⁵ Jubilee volume of the Central Institute for Meteorology, p. 329. Denkschriften, Imp. Acad. Science, Vol. XXIII. Vienna, 1901. See No. XXIII of this collection of translations.

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energy sufficient for the development of storms; hitherto it has only been assumed that storms start from such beginnings. The existence of unstable conditions before a storm has never been demonstrated. Prohaska never found such cases in his numerous studies of thunder-storms. Even in so-called calms there is still enough motion to disturb a condition of unstable equilibrium. The forces that are thus set free are greater than those corresponding to the largest horizontal pressure gradients that have been observed in the atmosphere. The accelerating force acting on a foreign particle of air whose temperature is T_1 when surrounded by air whose temperature is T , is

$$g \frac{(T_1 - T)}{T}$$

and therefore, for $T = 273^\circ$ and $T_1 = 274^\circ$ this becomes $\frac{g}{273}$, whereas

the force represented by a barometric gradient of 1^{mm} mercury per degree of a great circle, at the base of the atmosphere is $\frac{g}{1000}$. The

vertical distances in our atmosphere are small. Unstable conditions can scarcely exist for any length of time over extensive areas; they would disappear very quickly; their existence has not been demonstrated nor are they probable. Where there are adjacent masses of air with very large differences of temperature on the same level, as in the boe-en gusts, then cold air may intrude upon the warm region, and warm air flow over into the cold region, but in this case the storm wind velocity arises not only by reason of the barometric horizontal pressure but also directly by the action of gravity.

§(6) Many seek for the source of energy of a storm in the latent heat of condensation evolved by the formation of clouds. I will now compute the available kinetic energy for the following initial condition:

In chamber 1 of fig. 1a let there be dry air in neutral equilibrium, but in chamber 2 an ideal fictitious gas that has the property of expanding only when heat is added [and not by any diminution of pressure] but otherwise behaves like dry air. This latter gas replaces the moisture-saturated air [of nature] and the heat added during its expansion corresponds to the latent heat of condensation; for neutral equilibrium the vertical diminution of temperature in this fictitious gas is smaller than that in the dry air. The initial condition is to be so chosen that after the removal of the screen the mass 1 spreads out below and the mass 2 above; we may then omit

the assumption of neutral equilibrium for the result will hold good also when stable equilibrium exists initially in each chamber.

The equation of energy is now to be applied in the form

$$d\bar{K} + (\bar{R}) = (Q) - \delta(\bar{P} + \bar{I})$$

where (Q) replaces the latent heat of condensation. The mass 2 cools by expansion less than dry air, wherefore it contributes a smaller portion to the $-\delta(\bar{P} + \bar{I})$. The difference is exactly made up by the added heat (Q) . It follows that the latent heat of condensation contributes nothing to the energy of the storm. The available kinetic energy remains unchanged, if in the initial stage we substitute dry air of the same temperature for the fictitious gas. We must replace the moisture-saturated air with dry air of equal density, wherefore the latter having the same pressure must have a somewhat higher temperature.

§(7) The diagram, fig. 1, is not intended to give a complete idea of the phenomena in *boe-en*; it only contains that which we consider as the condition for the origin of the storm.

The length of the chamber must be measured by hundreds of kilometers but the altitude by very few kilometers. It is indifferent whether the separating surface (or screen) is initially vertical or inclined, if only the wedge-shaped volume is small relative to that of the whole chamber. If the [inclined] boundary surface be so laid that the colder mass of air 1 extends as a sharp wedge toward the ground, we have everywhere steady distribution of pressure. For an angle of 10° between the boundary surface and the horizon and for a chamber of 3000 meters altitude the length of the wedge is 17 kilometers; the diminution of pressure at the ground extends over this distance. If the boundary advances at the rate of 85 kilometers per hour, then in 12 minutes the barometer at any place will rise by the amount that corresponds to the difference in the weights of unit columns of cold and warm air. If the length of the chamber were 500 kilometers, the result would not be notably different but the computation would be more troublesome. Even an inclination of 1° for the boundary surface with a distribution of the fall of pressure over a distance of 170 kilometers would allow of the existence of a great amount of available kinetic energy.

I have as yet no definite idea as to how a condition involving the presence of a great store of potential energy arises without the immediate occurrence of an unloading or diminution of the potential by virtue of some movement of the air; in the present memoir

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we simply assume an initial stage generalized from actual observations.

The restriction of the mass of air to a closed system, the assumption of a well-defined boundary between the cold and the warm air, the introduction of level surfaces of equal pressure, are all analytical auxiliaries that we employ in order to ascertain the value of the available kinetic energy of a perfectly definite system.

§(8) If we are to give a broad interpretation to the results of our analyses, we must omit numerical details. The phenomena of motion in the great storm areas that we call cyclones are less intelligible than those of the boe-en. But these also, at least in median and higher latitudes, consist of warm and cold masses of air lying adjacent to each other horizontally; cold air often spreads out over the earth in the lower strata behind the passing storm. It is therefore not unlikely that these storms are fed by the potential energy of an initial stage similar to that which we have adopted in the preceding lines.

The opinion that the energy of cyclones arises from the overturning of masses of unequal temperature has been expressed in recent years by Prof. F. H. Bigelow. In one of his memoirs⁶ we find the following sentence:

"The cyclone is not formed from the energy of the latent heat of condensation, however much this may strengthen its intensity; it is not an eddy in the eastward drift, but is caused by the counterflow and overflow of currents of air of different temperatures."

Long ago W. Blasius had vainly labored to introduce a similar view as to the origin of storms. In his books (*Storms*, Philadelphia, 1875; and "*Stürme und Moderne Meteorologie*," Braunschweig, 1893) this idea is found mixed up with other views that are, I believe, less reasonable; however, they deserve careful consideration. From giving too much attention to the isobars, it has been generally assumed that the air must ascend only in the central part of a region of low pressure and that the motions are nearly symmetrical about this. Ferrel as well as Guldberg and Mohn have built on this assumption.

Our analysis gives us only a general idea as to the source of the energy of storms; a working model of the cyclone with symmetrical distribution of temperature has not yet been constructed.

⁶ "Studies on the meteorological effects of the solar and terrestrial physical processes. Weather Bureau publication, No. 290, Washington, 1903, p. 37; separate print from the *Monthly Weather Review*, Feb., 1903, p. 84, column 2.

It is stated of tropical whirlwinds that ordinarily within their areas there is observed at the earth's surface no great difference of temperature and that the cyclonic distribution of pressure only extends upward to altitudes of a few kilometers. Hence it may be concluded that near the ground the central part of the storm area is the warmest. Helmholtz, in his address on "Whirlwinds and Thunderstorms," assumes that the storm begins with an unstable condition above a small area; according to the computations of Reye, the vertical distribution of temperature around this central region may be stable; the lower strata are pushed upward through this central hole; the layers that at first were above then sink lower. This process is dependent materially on the vapor content; and on another occasion we will investigate how the available energy is to be computed for this case. In order to judge whether this idea will apply to tropical cyclones we must know accurately the conditions, especially the temperature both inside and outside the relatively small area of the storm.

W. M. Davis has shown⁷ that tornadoes originate only in the neighborhood of the boundary between cold and warm masses of air, a circumstance that had scarcely been considered before and the knowledge of which can be very useful in finding the source of energy of these enigmatical storms.

ADOPTED NOTATION

§(9) The following notation and the numerical values of the more important constants will be used:

t , the time.

$x y z$, the rectangular coördinates of a point referred to a system of axes that rotate with the earth, z being vertical and positive upwards

c , the velocity at this point relative to the origin of these axes.

$u v w$, the corresponding components of the velocity c .

k , the volume.

m , the mass.

$\frac{d}{dt}$, the symbol for the total differential coefficient of a function with regard to the time or

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$$

⁷ Hann: Lehrbuch der meteorologie, Braunschweig, 1901, p. 705.

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∂ , round d , the symbol for partial differentiation of any function or variable.

W , the potential of the attraction of gravitation combined with the centrifugal force of the diurnal rotation,

$g = \frac{\partial W}{\partial z}$ = the local apparent force of gravity, assumed to be constant in these present applications and to be $9.805966 \text{ m/sec.}^2$

A_x, A_y, A_z , rectangular components of the deflecting force due to the rotation of the earth, or the general terms of the equations of motion that must satisfy the condition

$$u A_x + v A_y + w A_z = 0.$$

p , elastic pressure within the air.

μ , density of the air.

T , absolute temperature of the air in Centigrade degrees.

T^* , average temperature of large mass of air.

$a = \frac{\partial T}{\partial z}$ = rate of change of temperature of atmosphere with altitude.

R , the constant of the equation of elasticity for gas.

$p = RT\mu$, the equation of elasticity for gas.

C_p , specific heat of air under constant pressure.

C_v , specific heat of air under constant volume.

$$C_p - C_v = R = \kappa C_p; \kappa = \frac{R}{C_p}; \gamma = \frac{C_p}{C_v}$$

For adiabatic change from $p_0 \mu_0 T_0$ to $p \mu T$ we have

$$\frac{p}{p_0} = \left(\frac{T}{T_0} \right)^{1/\kappa} = \left(\frac{\mu}{\mu_0} \right)^\gamma$$

For dry air the numerical values are

$$\gamma = 1.41; \kappa = \frac{0.41}{1.41}$$

$$R = 287.026 \frac{\text{met}^2}{\text{sec}^2 \text{ C}^0} = 0.0689 \frac{\text{Cal}}{\text{kg C}^0};$$

$$C_p = 987.09 \frac{\text{met}^2}{\text{sec}^2 \text{ C}^0} = 0.237 \frac{\text{Cal}}{\text{kg C}^0}$$

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$-\bar{R}$, work done by frictional forces or $+\bar{R}$ the kinetic energy lost by the whole system through friction.

$\bar{R}_x, \bar{R}_y, \bar{R}_z$, the corresponding rectilinear components.

$\bar{Q}$, quantity of heat added to unit mass of air per unit time.

S , entropy of the unit mass of air.

(Q) , quantity of heat added to the whole closed system in whole time.

V , the average velocity of motion throughout any closed system for the whole time.

$\bar{K}$, kinetic energy of the whole mass of air in the system.

$\delta\bar{K}$, increase of kinetic energy of the system.

$\bar{T}$, potential temperature of air whose actual temperature is T at pressure p .

$\bar{P}$, potential energy of the system due to its position and the action of gravity.

$\delta\bar{P}$, change in potential energy of the system accompanying the increase of kinetic energy, $\delta\bar{K}$.

$\bar{I}$, internal energy of the system.

$\delta\bar{I}$, change of internal energy of the system when pressural forces do external work.

$\delta\bar{A}$, the work done by pressural forces.

For the notation and values of these quantities in moist air see Chapter IV later.

CHAPTER I

THE EQUATIONS OF ENERGY OF A MOVING PARTICLE AND OF THE WHOLE MASS OF AIR IN A CLOSED SYSTEM

§(10) One of the equations that holds good for the motion of the air relative to the system of coördinates rotating with the earth is

$$\frac{du}{dt} = -\frac{\partial W}{\partial x} - \frac{1}{\mu} \frac{\partial p}{\partial x} + \bar{R}_x + A_x$$

and the other two are analogous. From these three there results as the equation of energy of a definite particle of air of unit mass moving with the velocity c ,

$$\frac{d}{dt} \left(\frac{c^2}{2} + W \right) + \frac{1}{\mu} \left(\frac{dp}{dt} - \frac{\partial p}{\partial t} \right) - \bar{R} c \cos(\bar{R}, c) = 0 \quad (1)$$

The increase of the total kinetic and potential energy of the particle is equal to the work done by the pressural forces and the frictional forces. The deflecting force of the rotation of the earth being normal to the path does no work; this is also true of the other portions of the term A . Equation (1) differs from the equation of energy for absolute motion only in respect to the meaning of W which contains a term depending on the rotation of the earth in addition to the potential of the attraction.

We combine the dynamical relation (1) with the thermal relation

$$\frac{dQ}{dt} = C_v \frac{dT}{dt} - \frac{1}{\mu} \frac{dp}{dt} \quad \dots \quad (2)$$

which applies when the quantity of heat dQ is imparted to the particle of air while describing its path ds during the element of time dt . It is here assumed that just as in air at rest, so here the imparted heat dQ serves only to increase the internal energy by the quantity $C_v dT$ and to perform the work of expansion $p d\left(\frac{1}{\mu}\right)$. Consequently

$$\frac{dQ}{dt} = C_v \frac{dT}{dt} + \frac{d}{dt} \left(\frac{c^2}{2} + W \right) - \frac{1}{\mu} \frac{p}{t} - \bar{R} c \cos(\bar{R} c) \quad \dots \quad (3)$$

We must include in dQ not only the heat communicated from without but also that portion of the heat due to friction that belongs to this small elementary unit mass of air.

§(11) The equations (1) and (3) with the factor μdk when integrated over the space k , assumed to be filled with air, give the relations for the total energy of the whole mass within that space.

In this integration we make use of the equation of continuity

$$\frac{\partial \mu}{\partial t} + \frac{\partial (\mu u)}{\partial x} + \frac{\partial (\mu v)}{\partial y} + \frac{\partial (\mu w)}{\partial z} = 0$$

If F indicates a quantity that is considered as attached to a definite elementary mass and can be expressed by a continuous function of the place and time, then by using a well-known transformation we have

$$\begin{aligned} \int \frac{dF}{dt} \mu dk &= \int \frac{\partial F}{\partial t} \mu dk + \int \left(\mu u \frac{\partial F}{\partial x} + \mu v \frac{\partial F}{\partial y} + \mu w \frac{\partial F}{\partial z} \right) dk \\ &= \frac{\partial}{\partial t} \int \mu F dk - \int \mu F c \cos(n, c) dO \end{aligned}$$

where n is the direction of the normal directed inward to the surface O of the space k .

In a similar way we obtain

$$\int \left(\frac{dF}{dt} - \frac{\partial F}{\partial t} \right) dk = \int \frac{F}{\mu} \frac{d\mu}{dt} dk - \int F c \cos(n, c) dO$$

The surface integrals disappear in our case when the mass of air is bounded by fixed walls; and so also when we extend the space integral or the mass integral over the whole atmosphere, or over a mass of air that is bounded by the ground and fixed vertical walls but is open above; assuming that both the pressure and also the product of pressure by vertical velocity diminish to zero as the altitude increases.

The equation for the energy of the whole mass of a closed system as obtained from equation (1) is as follows:

$$\frac{\partial}{\partial t} \int \left(\mu \frac{c^2}{2} + \mu W \right) dk + \int \frac{p}{\mu} \frac{d\mu}{dt} dk - \int \bar{R} c \cos(\bar{R}, c) \mu dk = 0. \quad (4)$$

We would remark that here in closed systems the work done by the pressural forces is equal to the sum of the work done by the expansions of all the elementary masses or

$$- \int \frac{1}{\mu} \left(\frac{dp}{dt} - \frac{\partial p}{\partial t} \right) \mu dk = \int p \frac{d}{dt} \left(\frac{1}{\mu} \right) \mu dk$$

In a similar way there follows from equation (2)

$$\left. \begin{aligned} \int \frac{dQ}{dt} \mu dk &= C_p \frac{\partial}{\partial t} \int T \mu dk - \int \frac{p}{\mu} \frac{d\mu}{dt} dk - \int \frac{\partial p}{\partial t} dk \\ &= C_v \frac{\partial}{\partial t} \int T \mu dk - \int \frac{p}{\mu} \frac{d\mu}{dt} dk \end{aligned} \right\} \quad (5)$$

From equations (4) and (5) there results the following equation (6) which we call the equation of energy for a mass of air in a closed system

$$\left. \begin{aligned} \int \frac{dQ}{dt} \mu dk &= C_v \frac{\partial}{\partial t} \int T \mu dk + \frac{\partial}{\partial t} \int \left(\mu \frac{c^2}{2} + \mu W \right) dk - \\ &\quad - \int \bar{R} c \cos(\bar{R}, c) \mu dk \end{aligned} \right\} \quad (6)$$

These express
the following

$$\bar{K} = \int \frac{\mu c^2}{2}$$

$$\bar{P} = \int \mu W$$

$$\bar{I} = C_v \int$$

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These expressions become more perspicacious by the introduction of the following abbreviating symbols:

$$\bar{K} = \int \frac{\mu c^2}{2} dk = \text{the kinetic energy of the whole mass of air in the closed system.}$$

$$\bar{P} = \int \mu W dk = \text{the potential energy of position.}$$

$$\bar{I} = C_v \int T \mu dk = \frac{C_p}{R} \int p dk = \text{the internal energy.}$$

The changes in the values of these quantities in the time t are indicated by $\partial \bar{K}$, $\partial \bar{P}$, $\partial \bar{I}$ and are completely determined by the initial and final conditions.

The three following quantities depend on the path that each elementary mass pursues; these time integrals extend over the same interval t .

$$-\partial \bar{A} = - \int dt \int \frac{p d\mu}{\mu dt} dk \dots \left\{ \begin{array}{l} \text{the work of the pressural forces,} \\ \text{or the work of expansion in the} \\ \text{time } t. \end{array} \right.$$

$$-(\bar{R}) = \int dt \int \bar{R} c \cos(\bar{R} c) \mu dk \left\{ \begin{array}{l} \text{the work of the frictional forces} \\ \text{or loss of energy, } \bar{R}, \text{ by friction,} \\ \text{in the time } t. \end{array} \right.$$

$$(Q) = \int dt \int \frac{dQ}{dt} \mu dk \dots \left\{ \begin{array}{l} \text{the quantity of heat communi-} \\ \text{cated to the closed system, in the} \\ \text{time } t. \end{array} \right.$$

We therefore write the equations for the kinetic energy, thermal equilibrium, and total energy of the whole mass, respectively as follows:

$$\text{Kinetic energy} \dots \partial(\bar{K} + \bar{P} + \bar{A}) + (\bar{R}) = 0 \dots (4^*)$$

$$\text{Thermal energy} \dots (Q) = \partial \bar{I} - \partial \bar{A} \dots (5^*)$$

$$\text{Total energy} \dots (Q) = \partial(\bar{K} + \bar{P} + \bar{I}) + (\bar{R}) \dots (6^*)$$

§(12) In the case of friction there can be no steady motion without a corresponding continual addition of heat. $\bar{K}$, $\bar{P}$, $\bar{I}$ remain unchanged in steady motion and the equations reduce to $(Q) = (\bar{R}) = -\partial \bar{A}$. The additions of heat (necessarily consisting of posi-

tive and negative portions), the loss of energy by friction and the work of the pressural forces are equal to each other.

$(Q) = 0$. Atmospheric motions are primarily dependent on the heat communicated from without. But it is possible that for certain general movements of the air, it is not the absorption of heat during their occurrence that is the determining factor but the temporary distribution of pressure, temperature and velocity throughout the mass of air. In the following pages we will treat these latter movements as though there were no addition of heat but only an effort to attain equilibrium as the result of some given initial condition.

It is a characteristic of every motion performed without exchange of heat that the work done by the pressural forces is determined by the initial and final stages alone. It is well known that the quantity $\bar{A}$ has the significance of a potential energy under certain conditions such as, when the mass is kept at constant temperature, or when every elementary particle of mass behaves adiabatically. This last condition is a special case of $(Q) = 0$. In this case an interchange of heat within the interior of the mass is allowable, or such additions and subtractions of heat as balance each other, or sum up zero, during the whole time t under consideration. If we confine ourselves to the case that $(Q) = 0$ during every element of time then the changes of $\bar{A}$ and $\bar{I}$ are continuously equal to each other. In the case of motion without increase of heat, the quantity $\bar{A}$ that I have in other places⁸ called the potential energy of the distribution of pressure becomes identical with the internal energy of the mass.

If $(Q) = 0$ then

$$\partial \bar{K} + (\bar{R}) = - \partial (\bar{P} + \bar{I}) \dots \dots \dots (6^{**})$$

In this case $\bar{P} + \bar{I}$ is to be considered as the total potential energy of the system; in the case of gaseous motions in a closed system this expression has the same meaning as $\bar{P}$ alone in the case of motions of rigid bodies or incompressible fluids under the influence of gravity. One easily recognizes the meaning of $\bar{I}$ in the expression for the potential energy if one considers the case of masses of air pushed horizontally (by compression or expansion) out of the condi-

⁸ Max Margulès: Ueber den Arbeitswert einer Luftdruckvertheilung und ueber die Erhaltung der Druckunterschiede. Jubiläum des KK Central Anstalt, Vienna. 1901. See No. XXIII of this collection of translations

tion of equilibrium while $\bar{P}$ remains unchanged. If the displacement of each element of mass proceeds adiabatically, or if there be an internal interchange of heat, then $\bar{I}$ must increase; after this if the mass is left to itself it strives toward the position of minimum $\bar{I}$.

The problem that we would solve by means of equation (6**) is as follows:

A mass of air in a closed system is at the beginning at rest and has a given initial internal distribution of temperature and pressure. It is set in motion by its tendency toward a condition of stable equilibrium. If there were no friction the individual portions of the mass would oscillate about their positions of equilibrium. In the presence of friction the final condition is attained by the gradual consumption of the kinetic energy. We seek the maximum values of $\delta\bar{K} + (R)$ which we designate as the available kinetic energy of the system. Since the initial values of $\bar{P}$ and $\bar{I}$ are known our problem is to compute their values for the final stage.

No special assumption will be introduced as to the frictional forces; in fact for turbulent motions there is no assumption that can satisfy all demands. Hence the frictional force will not be treated as a part of the pressural force, as is usual nowadays in the treatment of steady motions. The condition $(Q) = 0$ demands that the heat generated in the system by friction be immediately withdrawn. But this is not of great importance in atmospheric motions. Even in cases where it is assumed that the addition of heat has no influence on the motion there are more abundant sources of heat than this friction. For the present study the important point is that the heat due to friction shall not give rise to new or additional kinetic energy [during the interval of time under consideration].

CHAPTER II

APPLICATION OF THE EQUATION OF ENERGY TO THE OVERTURNING OF STRATA IN A COLUMN OF AIR

§(13). The analyses to be executed in this second section will not afford much that is new, they serve only as preparatory to the following sections.

Consider a column of air of unit area section, in which p and T

are functions of the altitude z only. The condition of hydrostatic equilibrium is

$$\left. \begin{aligned} \text{or} \quad & g + \frac{1}{\mu} \frac{\partial p}{\partial z} = 0 \\ & \frac{1}{p} \frac{\partial p}{\partial z} = - \frac{g}{RT} \end{aligned} \right\} \dots \dots \dots (\alpha)$$

The analysis is simplified by the assumption that g is independent of z whence the potential equation becomes

$$W = gz \dots \dots \dots (\beta)$$

Let us distinguish between a lower part of the column having a mass M extending to the altitude h and a higher mass M_h . Let overturnings of the strata take place in M whereby h is changed but the pressure

$$p_h = gM_h \dots \dots \dots (\gamma)$$

on the upper boundary of M remains constant.

Let Z be the altitude of the center of gravity of M_h then during all changes $Z - h$ remains unchanged, since M_h rises and falls like a solid piston.

The potential energy of position for the whole column of air is

$$\bar{P} = \int_0^\infty gz \mu dz = \int_0^h gz \mu dz + gM_h z$$

By utilizing the relation (α) we have

$$\begin{aligned} \int_0^h gz \mu dz &= - \int_{p_0}^{p_h} z dp = \int_0^h p dz - h p_h \\ \bar{P} &= \int_0^h p dz + (Z - h) p_h = R \int T dm + \text{Constant} \dots (1) \end{aligned}$$

where the integral in the last part of the equation is to be extended over the whole of the lower mass M .

The internal energy of the column is

$$\bar{I} = C_v \int T dm + \text{Constant} \dots \dots \dots (2)$$

Overturning of the strata can occur spontaneously whenever the

center of gravity sinks thereby; the available kinetic energy in the initial stage is

$$\delta \bar{K} + (\bar{R}) = -\delta (\bar{P} + \bar{I})$$

and for $(Q) = 0$ this becomes

$$-C_p \int \delta T \, dm,$$

which is in the ratio C_p/R or $1.41/0.41 = 3.44$ larger than the work done by gravity; the principal part of the kinetic energy is derived from the internal energy.

Let T and T' be the temperatures of the elementary mass dm and T^* and $T^{*'} the average temperatures of the whole mass M at the initial and final stages, respectively, then we have$

$$\delta \bar{K} + (\bar{R}) = C_p \int (T - T') \, dm = C_p (T^* - T^{*'}) M. \quad (3)$$

Under the condition here given, kinetic energy is available; that is to say, when by adiabatic overturning of the layers the average temperature of the whole mass of air sinks, then the condition is not stable.

§(14) *Computation of the available kinetic energy in the case of any change of position of a layer.*

Let the thin layer m_1 (see fig. 2) that initially lay beneath M_2 be adiabatically brought to lie above M_2 . In this case nothing changes in the lower mass M_0 since the upper mass M_2 acts like a piston of constant weight. Let the layers of the mass M_2 retain the same consecutive order. The kinetic energy that is available when the mass m_1 ascends in small particles and spreads out over M_2 , is now to be computed from equation (3).

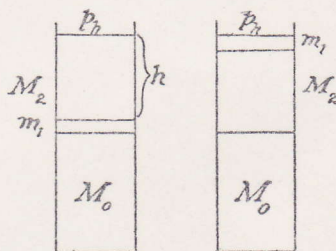


FIG. 2.

Let p_1 and T_1 be the initial pressure and temperature of m_1 ; p_h and T'_1 the same quantities at the end. Then

$$T'_1 = T_1 \left(\frac{p_h}{p_1} \right)^{\kappa}; \quad p_1 = p_h + g M_2$$

Let p_2 , T_2 and p'_2 , T'_2 be the corresponding pairs of values for a stratum dM_2 , then from the adiabatic condition and from

$p'_2 = p_2 + gm_1$ and from the assumption that gm_1 is small relative to p_2 there follow

$$T'_2 = T_2 \left(\frac{p'_2}{p_2} \right)^\kappa = T_2 \left(1 + \frac{gm_1}{p_2} \right)^\kappa = T_2 \left(1 + \kappa \frac{gm_1}{p_2} \right)$$

$$C_p (T_2 - T'_2) = -gm_1 \frac{RT_2}{p_2} = -\frac{gm_1}{\mu_2}$$

The contribution of the whole mass M_2 to the right-hand side of equation (3) is

$$C_p \int (T_2 - T'_2) dM_2 = -gm_1 \int \frac{dM_2}{\mu_2} = -gm_1 \int dz = -gm_1 h$$

where h is the altitude of the mass M_2 at the initial stage.

The overturning of the masses is brought about by the slightest initial impulse, and kinetic energy develops when the equation

$$-\delta(\bar{P} + \bar{I}) = m_1 \left\{ C_p T_1 \left[1 - \left(\frac{p_h}{p_1} \right)^\kappa \right] - gh \right\} \quad (4)$$

has a positive value.

$$\Delta h + \Delta gz$$

§(15) *Continuous distribution of temperature.* We will first assume that the separation between m_1 and M_2 was only made for the convenience of computation and that in the initial stage the temperature is a continuous function of the altitude throughout these masses. The value of h must differ from that appropriate to the final stable position, but it may be chosen small enough to allow us to assume T_2 expressible as a linear function, viz.,

$$T = T_2 = T_1 - az \quad a = -\frac{\partial T}{\partial z}$$

With this function of the temperature the equation α gives us

$$\frac{p_h}{p_1} = \left(\frac{T_1 - ah}{T_1} \right)^{\sigma/Ra}$$

whence assuming ah to be small relative to T_1 we get

$$\left(\frac{p_h}{p_1} \right)^\kappa = \left(1 - \frac{ah}{T_1} \right)^{\sigma/C_p a} = 1 - \frac{gh}{C_p T_1} + \frac{g(g - C_p a)}{C_p^2 T_1^2} \frac{h^2}{2}$$

By substituting this in equation (4), the linear term in h disappears and there remains

$$-\delta(\bar{P} + \bar{I}) = \frac{gm_1}{2T_1} h^2 \left(a - \frac{g}{C_p} \right) \dots \dots (4a)$$

This value is positive and gives the kinetic energy made available by the overturning of m_1 provided

$$a > \frac{g}{C_p} \text{ or } -\frac{\partial T}{\partial z} > \frac{g}{C_p}$$

In this case the equilibrium of the column of air is unstable.

If $a < g/C_p$ then with every adiabatic overturning of any layer there is associated an increase of the total potential energy $\bar{P} + \bar{I}$ and the equilibrium of the column of air is stable. In the limiting case $a = g/C_p$ and the equilibrium is neutral.

§(16) *Discontinuous distribution of temperature.* If at the boundary between m_1 and M_2 the temperature passes suddenly from T_1 to T_2 it will suffice to assume M_2 to be of an order of magnitude similar to m_1 whence we will now indicate it by m_2 . The interchange of positions of m_1 and m_2 gives us

$$-\delta(\bar{P} + \bar{I}) = C_p \{ m_1 (T_1 - T'_1) + m_2 (T_2 - T'_2) \}$$

$$T'_1 = T_1 \left(1 - \frac{gm_2}{p_1} \right)^\kappa$$

$$T'_2 = T_2 \left(1 + \frac{gm_1}{p_2} \right)^\kappa$$

hence when p is the pressure at the boundary and gm is small in comparison with p we have

$$-\delta(\bar{P} + \bar{I}) = m_1 m_2 \frac{gR}{p} (T_1 - T_2) \dots \dots (4b)$$

which corresponds to unstable equilibrium when the warmer mass lies below.

§(17) *The entropy as the criterion of stable equilibrium in a column of air of uniform constant constitution.*

The condition for stable equilibrium can be expressed very simply when we make use of the entropy. If P, Q, S_0 are respectively the

pressure, temperature, and entropy for a normal condition of the air, then for any other pair of values, p and T , the entropy for a unit mass is

$$S = S_0 + C_p \log \frac{T}{\theta} - R \log \frac{p}{P}$$

For a column at rest we have from equation (α)

$$\frac{\partial S}{\partial z} = \frac{C_p}{T} \frac{\partial T}{\partial z} - \frac{R}{p} \frac{\partial p}{\partial z} = \frac{C_p}{T} \left(\frac{\partial T}{\partial z} + \frac{g}{C_p} \right)$$

whence it follows that stable equilibrium exists when S increases with the altitude, but unstable when S diminishes with increasing altitude.

This result also holds good for sudden changes of temperature at special localities; in such cases $S_2 - S_1 = C_p \log \frac{T_2}{T_1}$; the entropy increases with the altitude when $T_2 > T_1$ and the warmer layer lies above the colder.

§(18) *Potential temperature.* Helmholtz and von Bezold* define the potential temperature $\bar{T}$ of a mass whose actual temperature is T and pressure p as being that temperature which the mass will attain when brought adiabatically to the normal pressure P . Hence we have

$$\bar{T} = T \left(\frac{P}{p} \right)^{\kappa} \quad dS = C_p \frac{d\bar{T}}{\bar{T}} \quad S = C_p \log \bar{T} + \text{Constant.}$$

The equilibrium is stable if the layers are arranged to succeed each other upward in the order of increasing potential temperature or increasing entropy.

§(19) *Buoyancy of an elementary mass of air that has a temperature different from that of its surroundings.*

In a mass of air at rest where T is a function of the altitude only, we will introduce a foreign particle of air m , which has the temperature θ_0 at the altitude z_0 but always the same pressure p as that of

*See the article by Dr. L. A. Bauer, reprinted as No. XXII of this collection.—C. A.

the surrounding air at the same altitude. Under adiabatic conditions its temperature in all positions is given by the equation

$$\theta = \theta_0 \left(\frac{p}{p_0} \right)^\kappa$$

whence

$$\frac{1}{\theta} \frac{d\theta}{dz} = \frac{\kappa}{p} \frac{dp}{dz}$$

Let us assume that during the motion of the particle m the pressure in the larger mass remains unchanged and that the equation (a) holds good so that we have

$$\frac{d\theta}{dz} = - \frac{g}{C_p} \frac{\theta}{T}$$

If $[\mu]$ is the density of m but μ that of the surrounding air then there is acting upward on m the accelerating force or buoyancy

$$g \left(\frac{\mu}{[\mu]} - 1 \right)$$

or, since the pressures are the same,

$$g \left(\frac{\theta}{T} - 1 \right)$$

If we consider m as an elementary mass moving without friction and express its vertical ordinate by z we then have the equation of motion

$$\frac{d^2z}{dt^2} = g \left(\frac{\theta}{T} - 1 \right) = - C_p \frac{d\theta}{dz} - g$$

Hence follows the equation of energy, assuming that m has no initial velocity at z_0

$$\begin{aligned} \frac{m}{2} \left(\frac{dz}{dt} \right)^2 &= m \left\{ C_p (\theta_0 - \theta) - g (z - z_0) \right\} \\ &= m \left\{ C_p \theta_0 \left(1 - \frac{p}{p_0} \right)^\kappa \right\} - g (z - z_0) \end{aligned}$$

This latter expression is identical with the right-hand side of equation 4 section 14, substituting m_1 for m , T_1 for θ_0 and h for $(z - z_0)$.

beautiful.

The deduction of the amount of the available kinetic energy for the whole system is rigorous as there given, but is inaccurate in the present article. The mass of air does not remain at rest when m moves through it; also, even if the energy of every particle of the mass is small, still the sum total may be of the same order of magnitude as the kinetic energy of m .

With this proviso I will quote some figures computed by these last equations in order to show how large the velocity may be that is produced by this force of buoyancy when the initial difference of temperature amounts to 10° .

When the acceleration is 0 the greatest absolute value of the velocity w will be attained at the altitude ζ , where $\theta = T$. Let the temperature of the larger mass of atmosphere at the initial position z_0 be $T_0 = 273^\circ \text{C.}$ and in general $T = T_0 - a(z - z_0)$; when $a = \frac{\partial T}{\partial z} = 0$ we obtain

$$\begin{cases} \text{for } \theta_0 = T_0 + 10^\circ, & \zeta - z_0 = 988.6 \text{ meters, } w = 18.8 \text{ m./sec.} \\ \text{for } \theta_0 = T_0 - 10^\circ, & \zeta - z_0 = -1025 \text{ meters, } w = -19.3 \text{ m./sec.} \end{cases}$$

Therefore the values of $\zeta - z_0$ and w are larger in proportion as the rate of vertical diminution of temperature throughout the mass of

atmosphere is larger; thus when $a = \frac{1}{2} \frac{g}{C_p} = \frac{\partial T}{\partial z}$ we have

$$\begin{cases} \text{for } \theta_0 = T_0 + 10^\circ, & \zeta - z_0 = 1942 \text{ meters, } w = 26.4 \text{ m./sec.} \\ \text{for } \theta_0 = T_0 - 10^\circ, & \zeta - z_0 = -2090 \text{ meters, } w = -27.4 \text{ m./sec.} \end{cases}$$

Similar computations for moist air are given by Reye in his "*Wirbelstürme*," Hanover, 1872, p. 227, etc.

§(20) *Computation of $\int T dm$ for linear vertical diminution of temperature in the column of atmosphere.*

Assuming $T = T_0 - az$ then for the state of rest [or hydrostatic equilibrium] the pressure is given by the integration of the equation (α), i. e.,

$$p = p_0 \left(\frac{T}{T_0} \right)^{g/Ra}$$

whence we deduce the following equation (3), that will often be used hereafter, for the integral of the product of the temperature by

$$P_{+I} = c_p \int_0^z T \mu dz = \frac{c_p}{g} \cdot \frac{1}{1+k} (p_0 T_0 - p T)$$

the mass in a vertical column of atmosphere whose section has the unit area and for which $p = RT\mu$

$$\left. \begin{aligned} \int_0^z T \mu dz &= \frac{1}{R} \int_0^z p dz \\ &= \frac{1}{Ra} p_0 T_0^{-a/Ra} \int_{T_0}^T T^{-a/Ra} dT \\ &= \frac{1}{g + Ra} (p_0 T_0 - p T) \end{aligned} \right\} \dots \dots (5)$$

For neutral equilibrium throughout the atmospheric column we have $a = g/C_p$ whence in this case

$$\int_0^z T \mu dz = \frac{1}{g} \cdot \frac{1}{1+k} (p_0 T_0 - p T) \dots \dots (5a)$$

$k = R/c_p$

§(21) *Overturning of upper strata in the column; computation of the available kinetic energy in this case.*

We will apply equation (3) to the following problem. In one column (see fig. 3) are at first two masses, 1 and 2, each in neutral equilibrium; the entropy of the lower mass 2 is higher than that of 1; between these two masses as a whole there exists initially unstable equilibrium, after the overturning and in the final stable condition the whole mass 1, which now becomes 1', rests below and 2 becomes 2' on top. In this overturning, the altitudes of both strata are changed and also the location of their

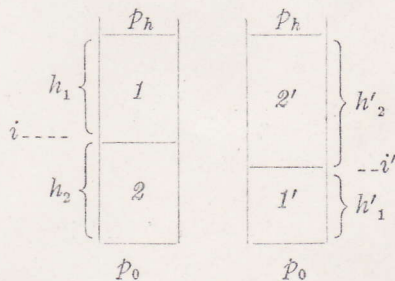


FIG. 3

surface of separation i ; the pressure on this surface i will be designated p_i for the initial and p'_i for the final stage; the temperature of each mass changes but the condition of neutral equilibrium remains true of 1' and 2' individually provided it be true as we assume that the transition from 1 to 1' and 2 to 2' is performed isentropically. The pressures p_0 at the base and p_h at the upper boundary remain unchanged.

At the boundary surface i there is initially a sudden transition from the temperature T_{i2} to the smaller value T_{i1} .

For this initial condition we have given the data

$$p_0, T_{02}, h_2, T_{i1}, h_1$$

whence we know

$$\begin{aligned} T_{i2} &= T_{02} - \frac{g h_2}{C_p} & T_{h1} &= T_{i1} - \frac{g h_1}{C_p} \\ p_i &= p_0 \left(\frac{T_{i2}}{T_{02}} \right)^{\frac{1}{\kappa}} & p_h &= p_i \left(\frac{T_{h1}}{T_{i1}} \right)^{\frac{1}{\kappa}} \end{aligned}$$

With these values by the help of equations (1), (2) and (5a) we compute the following equation except the arbitrary constant

$$(\bar{P} + \bar{I})_a = \frac{C_p}{g} \cdot \frac{2}{1 + \kappa} (p_0 T_{02} - p_i T_{i2} + p_i T_{i1} - p_h T_{h1}) + \text{Constant.}$$

For the final condition $p'_i = p_h + (p_0 - p_i)$ whence the temperatures of the two masses at the boundaries become

$$\begin{aligned} T'_{e1} &= T_{h1} \left(\frac{p_0}{p_h} \right)^{\kappa} & T'_{i1} &= T_{h1} \left(\frac{p'_i}{p_h} \right)^{\kappa} \\ T'_{i2} &= T_{02} \left(\frac{p'_i}{p_0} \right)^{\kappa} & T'_{h2} &= T_{02} \left(\frac{p_h}{p_0} \right)^{\kappa} \end{aligned}$$

and

$$(\bar{P} + \bar{I})_e = \frac{C_p}{g} \cdot \frac{1}{1 + \kappa} (p_0 T'_{01} - p'_i T'_{i1} + p'_i T'_{i2} - p_h T'_{h2}) + \text{Constant.}$$

We find the heights or depths of the masses 1' and 2' from

$$h'_1 = \frac{C_p}{g} (T'_{01} - T'_{i1}); \quad h'_2 = \frac{C_p}{g} (T'_{i2} - T'_{h2})$$

The sum of the two masses is $\frac{p_0 - p_h}{g}$ wherefore the available kinetic energy of a unit mass is

$$\frac{1}{2} V^2 = g \frac{(\bar{P} + \bar{I})_a - (\bar{P} + \bar{I})_e}{p_0 - p_h}$$

Numerical example. Let $h_1 = h_2 = 2000$ meters and let the other data for this problem be the numbers indicated by a star (*) as follows (the pressures being in millimeters of mercury and the temperatures absolute Centigrade):

Initial stage

$$\begin{array}{lll}
 p_0 = 760^* & p_i = 591.690^{\text{mm}} & p_h = 450.222^{\text{mm}} \\
 T_{02} = 283^* & T_{i2} = 263.1315^\circ & T_{i1} = 260.1315^\circ \quad T_{h1} = 240.2630^\circ \\
 h_1^* = 2000 \text{ meters} & & h_2^* = 2000 \text{ meters}
 \end{array}$$

Final stage

$$\begin{array}{lll}
 p_0 = 760^{\text{mm}} & p_i' = 618.532^{\text{mm}} & p_h = 450.222^{\text{mm}} \\
 T'_{01} = 279.773^\circ & T'_{i1} = 263.509^\circ & T'_{i2} = 266.548^\circ \quad T'_{h2} = 243.034^\circ \\
 h'_1 = 1637.17 \text{ meters} & & h'_2 = 2366.97 \text{ meters}
 \end{array}$$

$$(\bar{P} + \bar{I})_a = \frac{C_p}{g} \cdot \frac{1}{1 + \kappa} \cdot p_h \cdot 233.5132 + \text{Constant}$$

$$(\bar{P} + \bar{I})_e = \frac{C_p}{g} \cdot \frac{1}{1 + \kappa} \cdot p_h \cdot 233.4140 + \text{Constant}$$

$$\frac{1}{2} V^2 = \frac{p_h}{p_0 - p_h} \cdot \frac{1}{1 + \kappa} \cdot C_p 0.0992^\circ$$

whence

$$V = 14.85 \text{ meters per second.}$$

This computation is laborious because the conclusion depends on a small difference between large quantities. We may find V by a shorter method without previously computing a number of other quantities as in the next following article.

§(22) *Approximate method of computing V .* We first assume that each of the masses 1 and 2 is initially in stable equilibrium but that the greatest entropy in 1 is smaller than the least entropy in 2, so that as before in the final stage the whole of 1' comes to lie below 2'; the serial order of the strata in each of the masses will be retained provided the overturning proceeds isentropically.

Let p_1 and p'_1 be the pressures for the same layer in 1 and 1', or in the initial and final stages of mass 1 respectively, and similarly T_1 and T'_1 for the corresponding temperatures. Let p_2 ; p'_2 ; T_2 ;

T'_2 ; be the analogous quantities for any given layer in the mass 2 and 2'. We thus have

$$\begin{aligned} p'_1 &= p_1 + p_0 - p_i & p'_2 &= p_2 - (p_i - p_h) \\ T'_1 &= T_1 \left(\frac{p'_1}{p_1} \right)^\kappa = T_1 \left(1 + \frac{p_0 - p_i}{p_1} \right)^\kappa \\ &= T_1 \left(1 + \kappa \frac{p_0 - p_i}{p_1} \right) \text{ approximately.} \end{aligned}$$

With this last approximate value, which holds good in proportion as $p_0 - p_h$ is small relative to p_h , we obtain for mass 1

$$\begin{aligned} T_1 - T'_1 &= - \frac{p_0 - p_i}{C_p \mu_1} \\ C_p \int (T_1 - T'_1) dm_1 &= - (p_0 - p_i) h_1. \end{aligned}$$

Similarly for the other mass 2 we obtain

$$\begin{aligned} T_2 - T'_2 &= \frac{p_i - p_h}{C_p \mu_2} \\ C_p \int (T_2 - T'_2) dm_2 &= (p_i - p_h) h_2. \end{aligned}$$

Hence when h_1 and h_2 are small enough to allow of this approximate method we have

$$C_p \int (T - T') dm = (p_i - p_h) h_2 - (p_0 - p_i) h_1$$

From stable equilibrium we now pass on to the limiting case of neutral equilibrium within the masses 1 and 2; we have

$$p_h = p_i \left(1 - \frac{gh_1}{C_p T_{i1}} \right)^{1/\kappa} \quad \text{and} \quad p_0 = p_i \left(1 + \frac{gh_2}{C_p T_{i2}} \right)^{1/\kappa}$$

whence approximately

$$\begin{aligned} p_i - p_p &= p_i \frac{gh_1}{RT_{i1}} \quad \text{and} \quad p_0 - p_i = p_i \frac{gh_2}{RT_{i2}} \\ C_p \int (T - T') dm &= h_1 h_2 p_i \frac{g}{R} \cdot \frac{T_{i2} - T_{i1}}{T_{i1} T_{i2}} = \frac{1}{2} MV^2 \\ M &= \frac{p_0 - p_h}{g} = \frac{p_i}{R} \cdot \frac{h_1 T_{i2} + h_2 T_{i1}}{T_{i1} T_{i2}} \\ V^2 &= 2g \cdot \frac{h_1 h_2 (T_{i2} - T_{i1})}{h_1 T_{i2} + h_2 T_{i1}} \end{aligned}$$